# THE LJAPUNOV-SCHMIDT REDUCTION FOR SOME CRITICAL PROBLEMS

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## 1. INTRODUCTION

Let us consider the problem

$$\begin{cases} -\Delta u = |u|^{q-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$ and $q > 1$. Let $2^*$ denote the critical exponent in the Sobolev embeddings, i.e. $2^* = \frac{2n}{n-2}$.

In the subcritical case, i.e. $q < 2^* - 1$ compactness of Sobolev's embedding ensures existence of at least one positive solution and infinitely many sign changing solutions to (1).

In the critical case or in the supercritical case, i.e. $q \geq 2^* - 1$ existence of solutions is a delicate issue. In [39] Pohožaev proved that the problem (1) does not admit a nontrivial solution if $\Omega$ is star-shaped. On the other hand, Kazdan and Warner in [29] proved that problem (1) has one positive radial solution and infinitely many sign changing radial solutions if $\Omega$ is an annulus. In the critical case, i.e. $q = 2^* - 1$, Bahri and Coron in [2] found a positive solution to (1) provided the domain $\Omega$ has a *nontrivial topology.*

In this survey we are in particular interested in the following perturbed critical problems.

- *The Brezis-Nirenberg problem*

$$(\mathcal{BN})_\epsilon \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u + \epsilon u & \text{in } \Omega \subset \mathbb{R}^n, \ n \geq 4, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- *The "almost-critical" problem*

$$(\mathcal{AC})_\epsilon \qquad \begin{cases} -\Delta u = |u|^{2^*-2-\epsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- *The Coron's problem*

$$(\mathcal{C})_\epsilon \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega_\epsilon := \Omega \setminus B(z_0, \epsilon), \\ u = 0 & \text{on } \partial\Omega_\epsilon \end{cases}$$

In the first two problems $\epsilon \in \mathbb{R}$ is a small parameter either positive or negative. In the last problem $z_0 \in \Omega$ and $\epsilon$ is a small positive parameter.

The common feature of those problems is that when $\epsilon$ is small enough they can have solutions $u_\epsilon$ whose shape resembles the sum of a finite number of bubbles as $\epsilon$ goes to zero, i.e.

$$u_\epsilon(x) \sim \sum_{i=1}^k \lambda_i U_{\delta_i^\epsilon, z_i^\epsilon}(x)$$

where the concentration points $z_i^\epsilon$ converge to a point $z_i^0$ in $\Omega$ and the concentration parameters $\delta_i^\epsilon$ go to zero as $\epsilon$ go to zero. If $\lambda_i = +1$ ($\lambda_i = -1$) we say that $u_\epsilon$ has a positive (negative) blow-up point at $z_i^0$ as $\epsilon$ goes to zero.

A *bubble* is a function

$$U_{\delta,z}(x) := \alpha_n \frac{\delta^{\frac{n-2}{2}}}{\left(\delta^2 + |x-z|^2\right)^{\frac{n-2}{2}}}, \qquad \delta > 0, \quad x, z \in \mathbb{R}^n. \tag{2}$$

Here $\alpha_n := [n(n-2)]^{\frac{n-2}{4}}$. They are positive solutions to the limit problem (see Aubin [1], Caffarelli-Gidas-Spruck [11], Talenti [44])

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n. \tag{3}$$

The location of the points $z_i$'s where blowing-up occurs is strictly related to the geometry of the domain, namely Green's and Robin's functions. Let $G$ the Green's function of the negative laplacian on $\Omega$ with Dirichlet boundary conditions and let $H$ its regular part, i.e.

$$H(x,y) = \frac{c_n}{|x-y|^{n-2}} - G(x,y), \quad \forall (x,y) \in \Omega^2,$$

where $c_n$ is a positive constant. The function $\tau(x) := H(x,x)$, $x \in \Omega$ is called *Robin's function*. It is known that $\tau$ is a $C^2$−function and also that $\tau(x)$ goes to $+\infty$ as $x$ approaches the boundary of $\Omega$. Therefore, the Robin's function has always a minimum point in $\Omega$.

Let us state the main results concerning solutions to $(\mathcal{BN})_\epsilon$, $(\mathcal{AC})_\epsilon$ and $(\mathcal{C})_\epsilon$ which blow-up at one or more points of the domain as the parameter $\epsilon$ goes to zero.

*The Brezis-Nirenberg problem and the "almost critical" problem when $\epsilon$ is positive*

Brezis and Nirenberg in [10] proved that if $n \geq 4$ for small enough problem $(\mathcal{BN})_\epsilon$ has a positive solution provided $\epsilon$ is small enough. On the other hand, it is clear that the slightly sub-critical problem $(\mathcal{AC})_\epsilon$ has always a positive solution. Han in [27] proved that these solutions blow-up at a critical point of the Robin's function as $\epsilon$ goes to zero. Conversely, Rey in [40, 41] proved that any $C^1$−stable critical point $z_0$ of the Robin's function generates a family of solutions which blows-up at $z_0$ as $\epsilon$ goes to zero. Musso-Pistoia in [35] and Bahri-Li-Rey in [3] studied existence of solutions which blow-up at $\kappa$ different points of $\Omega$. Grossi-Takahashi [26] proved the nonexistence of positive solutions blowing up at $\kappa \geq 2$ points for these problems in convex domains.

As far as it concerns the existence of sign changing solutions, the slightly sub-critical problem $(\mathcal{AC})_\epsilon$ has infinitely many sign changing solutions. Existence of sign changing solution for problem $(\mathcal{BN})_\epsilon$ is a more difficult problem. The first result about problem $(\mathcal{BN})_\epsilon$ is due to Cerami-Solimini-Struwe, who showed in [13] the existence of a pair of least energy sign changing solutions if $n \geq 6$ and $\epsilon$ is small enough. The existence of infinitely many solutions to $(\mathcal{BN})_\epsilon$ for any $\epsilon > 0$ was established by Devillanova-Solimini in [22] when $n \geq 7$. Moreover, for low dimensions $n = 4, 5, 6$, in [23] they proved the existence of at least $n+1$ pairs of solutions provided $\epsilon$ is small enough. Ben Ayed-El Mehdi-Pacella in [7, 8] studied the blow up of the low energy sign-changing solutions of problems $(\mathcal{AC})_\epsilon$ and $(\mathcal{BN})_\epsilon$ as $\epsilon$ goes to zero and they classified these solutions according to the concentration speeds of the positive and negative part. In [12] Castro-Clapp proved the existence of one pair of solutions in a symmetric domain which change sign exactly once, provided $n \geq 4$ and $\epsilon$ is small enough. Moreover they describe the profile of the solutions, by showing that the solutions blow-up positively and negatively at two different points in $\Omega$ as $\epsilon$ goes to 0. Micheletti-Pistoia in [31] and Bartsch-Micheletti-Pistoia in [4] generalized such a result showing the existence of at least $n$ pairs of sign changing solutions with one negative and one positive blow-up points. Pistoia-Weth in [38] and Musso-Pistoia in [36] proved that a large number of sign changing solutions exists for problem $(\mathcal{AC})_\epsilon$: the solutions are a superposition with alternating sign of bubbles whose centers collapse to the minimum point of the Robin's function as $\epsilon$ goes to zero. This result is unknown for problem $(\mathcal{BN})_\epsilon$ even if we think to be true.

*The Brezis-Nirenberg problem and the "almost critical" problem when $\epsilon$ is negative*

In [39] Pohožaev proved that problems $(\mathcal{B})_\epsilon$ and $(\mathcal{AC})_\epsilon$ do not have any solutions if $\epsilon$ is negative and $\Omega$ is starshaped.

As far as it concerns the existence of blowing-up solutions, when $\epsilon$ is negative and small enough completely different phenomena take place even if the domain $\Omega$ is not starshaped. Indeed, Ben Ayed-El Mehdi-Grossi-Rey in [6] proved that problem $(\mathcal{AC})_\epsilon$ do not have any positive solutions which blows-up at one point when $\epsilon$ goes to zero. We believe that their argument could also be extended to the problem $(\mathcal{BN})_\epsilon$. Del Pino-Felmer-Musso in [21] and Musso-Pistoia in [32] found, for $\epsilon$ small enough, a positive solutions with two positive blow-up points provided the domain $\Omega$ has a hole. Del Pino-Felmer-Musso in [19, 20] and Pistoia-Rey in [37] found solutions with three or more positive blow-up points, under suitable assumptions on the domain $\Omega$. Towers of positive bubbles were constructed by del Pino-Dolbeault-Musso in [17, 18] and by Ge-Jing-Pacard in [25], under suitable assumptions on non degeneracy of Robin's and Green's functions. As far as it concerns the study of sign changing solutions, Ben Ayed-Bouh in [5] proved that problem $(\mathcal{AC})_\epsilon$, does not have any sign changing solutions with one positive and one or two negative blow-up points. We believe that their argument could also be extended also to the problem $(\mathcal{BN})_\epsilon$. There are no results about existence of sign changing solutions for these problems.

*The Coron's problem* Coron in [16] found via variational methods a solution to problem $(\mathcal{C})_\epsilon$ provided $\epsilon$ is small enough. If the domain has several holes, Rey in [42] and Li-Yan-Yang in [30] constructed solutions blowing-up at the centers of the holes as the size of the holes goes to zero. On the other hand, Clapp-Weth in [15] found a second solution to $(\mathcal{C})_\epsilon$, but they were unable to say if it was positive or changed sign. Clapp-Musso-Pistoia in [14] found positive and sign changing solutions to $(\mathcal{C})_\epsilon$ blowing-up at the center of the hole and at one or more points inside the domain as $\epsilon$ goes to zero. If the domain has two small holes, Musso-Pistoia in [34] constructed a sign changing solution with one positive blow-up point and one negative blow-up point at the centers of the two holes. Musso-Pistoia in [33] and Ge-Musso-Pistoia in [24] found a large number of sign changing solutions to $(\mathcal{C})_\epsilon$: the solutions are a superposition of bubbles with alternating sign whose centers collapse to the center of the hole as $\epsilon$ goes to zero.

The proofs of all the results concerning existence of solutions which blow-up positively or negatively at one or more points as the parameter $\epsilon$ goes to zero, rely on a Lyapunov-Schmidt reduction scheme firstly developed by Bahri-Coron in [2]. This allows to reduce the problem of finding blowing-up solutions to the problem of finding critical points of a functional which depends only on the blow-up points and the concentration rates. The leading part of the reduced functional is explicitly given in terms of the geometry of the domain, namely Green's and Robin's functions. The reduced functional also takes into account the different interactions among the bubbles which depends on their respective sign. Finally, we use a variational approach and we obtain the existence of critical points of the reduced functional by applying a minimization argument or a min-max argument. In the following we describe the main steps to get some of the previous results. We will refer to [35] and [3, 21, 31] for the proofs related to the construction of positive and sign-changing multi-bubbles to problems $(\mathcal{BN})_\epsilon$ and $(\mathcal{AC})_\epsilon$, respectively. We will refer to [36] and to [33, 24] for the proofs related to the construction of towers of bubbles to problems $(\mathcal{AC})_\epsilon$ and $(\mathcal{C})_\epsilon$, respectively.

## 2. SETTING OF THE PROBLEM

We want to rewrite problems $(\mathcal{BN})_\epsilon$, $(\mathcal{AC})_\epsilon$ and $(\mathcal{C})_\epsilon$ in a different, but equivalent, form.

Let us take

$$(u, v) := \int_\Omega \nabla u \cdot \nabla v \ dx, \qquad \|u\| := \left( \int_\Omega |\nabla u|^2 \ dx \right)^{1/2},$$

as the inner product in $H^1_0(\Omega)$ and its corresponding norm. Similarly, for each $r \in [1, \infty)$,

$$\|u\|_r := \left( \int_\Omega |u|^r \ dx \right)^{1/r}$$

is a norm in $L^r(\Omega)$.

Let $i^* : L^{\frac{2n}{n+2}}(\Omega) \to H^1_0(\Omega)$ be the adjoint operator to the embedding $i : H^1_0(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, i.e. $i^*(u) = v$ if and only if

$$(v, \varphi) = \int_\Omega u(x)\varphi(x)dx \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

if and only if

$$-\Delta v = u \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

It is clear that $i^*$ is a continuous map, namely there exists a positive constant $c$ such that

$$\|i^*(u)\| \le c \|u\|_{\frac{2n}{n+2}} \quad \forall \ u \in L^{\frac{2n}{n+2}}(\Omega).$$

To study the slightly supercritical case, namely problem $(\mathcal{AC})_\epsilon$ with $\epsilon < 0$, we need to find solutions in the space $H^1_0(\Omega) \cap L^{s_\varepsilon}(\Omega)$ with $s_\epsilon := \frac{2n}{n-2} - \epsilon \frac{n}{2}$. Indeed, by a well known Hardy-Littlewood-Sobolev inequality (see Hardy-Littlewood [28] and Sobolev [45]) we deduce that $i^*$ restricted to $H^1_0(\Omega) \cap L^{s_\varepsilon}(\Omega)$ is a continuous map, namely

$$\|i^*(u)\|_s \le c \|u\|_{\frac{s_\epsilon}{\frac{n+2}{n-2}-\epsilon}}$$

for some positive constant $c$ which depends only on $n$. We point out that if $\epsilon > 0$ then $H^1_0(\Omega) \cap L^{s_\varepsilon}(\Omega)$ coincides with $H^1_0(\Omega)$.

Using the above definitions and notations, it is clear that our problems can be rewritten in the equivalent form

$$\begin{cases} u = i^* \left[ f_\epsilon(u) \right] \\ u \in \mathfrak{H}_\epsilon, \end{cases} \tag{4}$$

where

(i) *for problem* $(\mathcal{BN})_\epsilon$

$$f_\epsilon(u) := |u|^{p-1}s + \epsilon u \quad \text{and } \mathfrak{H}_\epsilon := H^1_0(\Omega)$$

(ii) *for problem* $(\mathcal{AC})_\epsilon$

$$f_\epsilon(u) := |u|^{p-1-\epsilon}u \quad \text{and } \mathfrak{H}_\epsilon := H^1_0(\Omega) \cap L^{s_\varepsilon}(\Omega)$$

(iii) *for problem* $(\mathcal{C})_\epsilon$

$$f_\epsilon(u) := |u|^{p-1}u \quad \text{and } \mathfrak{H}_\epsilon := H^1_0(\Omega_\epsilon).$$

## 3. THE LJAPUNOV-SCHMIDT PROCEDURE

3.1. **The approximating solution.** The first step is writing a *good approximating solution.*

Let $PW$ denote the projection of the function $W \in D^{1,2}(\mathbb{R}^n)$ onto $\mathrm{H}_0^1(D)$, i.e.

$$\Delta PW = \Delta W \quad \text{in } D, \qquad PW = 0 \quad \text{on } \partial D,$$

where $D$ is a smooth bounded domain in $\mathbb{R}^n$.

Let $\kappa \geq 1$ be a fixed integer. We look for solutions $u_\epsilon$ to problems $(\mathcal{BN})_\epsilon$, $(\mathcal{AC})_\epsilon$ and $(\mathcal{C})_\epsilon$ as

$$u_\epsilon(x) = V_{\mathbf{z},\mathbf{d}}(x) + \phi(x), \quad V_{\mathbf{z},\mathbf{d}}(x) := \sum_{i=1}^{\kappa} \lambda_i P U_{\delta_i, z_i}(x), \tag{5}$$

where the higher order term $\phi$ belongs to a suitable space described in the next subsection. Here $\lambda_i \in \{-1, +1\}$, the concentration points $z_i$'s lye in $\Omega$ and the concentration parameters $\delta_i$'s are choose as follows.

- *Multi-bubbles for problem $(\mathcal{BN})_\epsilon$*

$$\mathfrak{M} - (\mathcal{BN})_\epsilon \qquad \begin{cases} \lambda_i \in \{-1, +1\} \\ z_1, \ldots, z_\kappa \in \Omega \text{ and } z_i \neq z_j \\ \delta_i = |\epsilon|^{\frac{1}{n-4}} d_i \quad \text{with} \quad d_i > 0 \end{cases}$$

The configuration space is

$$\Lambda := \{(\mathbf{z}, \mathbf{d}) : \mathbf{z} = (z_1, \ldots, z_\kappa) \in \Omega^\kappa, \ z_i \neq z_j, \ \mathbf{d} = (d_1, \ldots, d_\kappa) \in (0, +\infty)^\kappa\}.$$

- *Multi-bubbles for problem $(\mathcal{AC})_\epsilon$*

$$\mathfrak{M} - (\mathcal{AC})_\epsilon \qquad \begin{cases} \lambda_i \in \{-1, +1\} \\ z_1, \ldots, z_\kappa \in \Omega \quad \text{and} \quad z_i \neq z_j \\ \delta_i = |\epsilon|^{\frac{1}{n-2}} d_i \quad \text{with} \quad d_i > 0. \end{cases}$$

The configuration space is

$$\Lambda := \{(\mathbf{z}, \mathbf{d}) : \mathbf{z} = (z_1, \ldots, z_\kappa) \in \Omega^\kappa, \ z_i \neq z_j, \ \mathbf{d} = (d_1, \ldots, d_\kappa) \in (0, +\infty)^\kappa\}.$$

- *Tower of bubbles with alternating sign for problem $(\mathcal{AC})_\epsilon$ when $\epsilon > 0$*

$$\mathfrak{T} - (\mathcal{AC})_\epsilon \qquad \begin{cases} \lambda_i = (-1)^i \\ z_i = z + \delta_i \sigma_i \in \Omega \quad \text{with} \quad \sigma_1, \ldots, \sigma_{\kappa-1} \in \mathbb{R}^n \text{ and } \sigma_\kappa = 0 \\ \delta_i = \epsilon^{\frac{2(i-1)+1}{N-2}} d_i \quad \text{with } d_i > 0. \end{cases}$$

The configuration space is

$$\Lambda := \left\{(\mathbf{z}, \mathbf{d}) : \mathbf{z} = (\sigma_1, \ldots, \sigma_{\kappa-1}, z) \in \mathbb{R}^{(\kappa-1)n} \times \Omega, \ \mathbf{d} = (d_1, \ldots, d_\kappa) \in (0, +\infty)^\kappa\right\}.$$

- *Tower of bubbles with alternating sign for problem $(\mathcal{C})_\epsilon$*

$$\mathfrak{T} - (\mathcal{C})_\epsilon \qquad \begin{cases} \lambda_i = (-1)^i \\ z_i = z_0 + \delta_i \sigma_i \in \Omega \quad \text{with} \quad \sigma_1, \ldots, \sigma_k \in \mathbb{R}^N \\ \delta_i = \epsilon^{\frac{2i-1}{2k}} d_i \quad \text{with } d_i > 0. \end{cases}$$

The configuration space is

$$\Lambda := \{(\mathbf{z}, \mathbf{d}) : \mathbf{z} = (\sigma_1, \ldots, \sigma_\kappa) \in \mathbb{R}^{\kappa n}, \ \mathbf{d} = (d_1, \ldots, d_\kappa) \in (0, +\infty)^\kappa\}.$$

To say that $V_{\mathbf{z},\mathbf{d}}$ is a *good approximating solution* we need to estimate the error

$$\mathfrak{R}_{\mathbf{z},\mathbf{d}} := V_{\mathbf{z},\mathbf{d}} - i^* [f_\epsilon(V_{\mathbf{z},\mathbf{d}})] \in \mathfrak{H}_\epsilon.$$

**Proposition 3.1.** *For any compact subset* **C** *of* $\Lambda$ *there exist* $\epsilon_0 > 0$ *and* $c > 0$ *such that for each* $\epsilon \in (0, \epsilon_0)$ *and* $(\mathbf{z}, \mathbf{d}) \in \mathbf{C}$ *we have*

$$\|\mathfrak{R}_{\mathbf{z},\mathbf{d}}\| \leq c|\epsilon|^\eta.$$

*for some* $\eta > 0$ *which depends only on* $n$ *and* $\kappa$.

3.2. **The equation becomes a system.** The second step is writing the equation as a system.

We need to fix the space where the rest term $\phi$ in (5) belongs to. It is important to introduce the functions

$$\psi^0_{\delta,z}(x) := \frac{\partial U_{\delta,z}}{\partial \delta} = \alpha_n \frac{n-2}{2} \delta^{\frac{n-4}{2}} \frac{|x-z|^2 - \delta^2}{(\delta^2 + |x-z|^2)^{n/2}}$$

and, for each $j = 1, \ldots, n$,

$$\psi^j_{\delta,z}(x) := \frac{\partial U_{\delta,z}}{\partial z_j} = \alpha_n (n-2) \delta^{\frac{n-2}{2}} \frac{x_j - z_j}{(\delta^2 + |x-z|^2)^{n/2}},$$

which span the set of solutions to the linearized problem (see Bianchi-Egnell [9])

$$-\Delta \psi = f'_0(U_{\delta,z}) \psi \text{ in } \mathbb{R}^n.$$

Remember that the $\delta_i$'s, the $z_i$'s and the configuration space $\Lambda$ are given in $\mathfrak{M} - (\mathcal{BN})_\epsilon$, $\mathfrak{M} - (\mathcal{AC})_\epsilon$, $\mathfrak{T} - (\mathcal{AC})_\epsilon$ and $\mathfrak{T} - (\mathcal{C})_\epsilon$.

If $(\mathbf{z}, \mathbf{d}) \in \Lambda$, we introduce the spaces

$$K_{\mathbf{z},\mathbf{d}} := \mathrm{span}\{P\psi^j_{\delta_i,z_i} \ : \ i = 1, \ldots, \kappa, \ j = 0, 1, \ldots, n\},$$

$$K^\perp_{\mathbf{z},\mathbf{d}} := \left\{ \phi \in \mathfrak{H}_\epsilon \ : \ (\phi, P\psi^j_{\delta_i,z_i}) = 0, \ i = 1, \ldots, \kappa, \ j = 0, 1, \ldots, n \right\}$$

and the projection operators

$$\Pi_{\mathbf{z},\mathbf{d}}(u) := \sum_{i=1}^\kappa \sum_{j=0}^n (u, P\psi^j_{\delta_i,z_i}) P\psi^j_{\delta_i,z_i} \quad \text{and} \quad \Pi^\perp_{\mathbf{z},\mathbf{d}}(u) := u - \Pi_{\mathbf{z},\mathbf{d}}(u).$$

Our approach to solve problem (4) will be to solve the system

$$\begin{cases} \Pi^\perp_{\mathbf{z},\mathbf{d}} \left\{ V_{\mathbf{z},\mathbf{d}} + \phi - i^* \left[ f_\epsilon(V_{\mathbf{z},\mathbf{d}} + \phi) \right] \right\} = 0, \\ \Pi_{\mathbf{z},\mathbf{d}} \left\{ V_{\mathbf{z},\mathbf{d}}, + \phi - i^* \left[ f_\epsilon(V_{\mathbf{z},\mathbf{d}}, + \phi) \right] \right\} = 0, \\ (\mathbf{z}, \mathbf{d}) \in \Lambda \quad \text{and} \quad \phi \in K^\perp_{\mathbf{z},\mathbf{d}}. \end{cases} \tag{6}$$

3.3. **The reduction argument.** The third step is reducing the problem to a finite dimensional one.

The first result we need concerns the invertibility of the linear operator $\mathfrak{L}_{\mathbf{z},\mathbf{d}} : K^\perp_{\mathbf{z},\mathbf{d}} \to K^\perp_{\mathbf{z},\mathbf{d}}$ defined by

$$\mathfrak{L}_{\mathbf{z},\mathbf{d}}\phi := \phi - \Pi^\perp_{\mathbf{z},\mathbf{d}}, i^* \left[ f'_0(V_{\mathbf{z},\mathbf{d}})\phi \right].$$

We will prove the following.

**Proposition 3.2.** *For any compact subset* **C** *of* $\Lambda$ *there exist* $\epsilon_0 > 0$ *and* $c > 0$ *such that for each* $\epsilon \in (0, \epsilon_0)$ *and* $(\mathbf{z}, \mathbf{d}) \in \mathbf{C}$ *the operator* $L_{\mathbf{z},\mathbf{d}}$ *is invertible and*

$$\|L_{\mathbf{z},\mathbf{d}}\phi\| \geq c\|\phi\| \qquad \forall \ \phi \in K^\perp_{\mathbf{z},\mathbf{d}}.$$

Secondly, we solve the first equation in system (6), namely for each $(\mathbf{z}, \mathbf{d}) \in \Lambda$ and small $\epsilon$ we find a function $\phi \in K^\perp_{\mathbf{z},\mathbf{d}}$ which solves the first equation in system (6). To do this, we use a simple contraction mapping argument together with the estimate of the rest term given in Proposition 3.1.

**Proposition 3.3.** *For any compact subset* $\mathbf{C}$ *of* $\Lambda$ *there exist* $\epsilon_0 > 0$ *and* $c > 0$ *such that for each* $\epsilon \in (0, \epsilon_0)$ *and* $(\mathbf{z}, \mathbf{d}) \in \mathbf{C}$ *there exists a unique* $\phi_{\mathbf{z},\mathbf{d}}^\epsilon \in K_{\mathbf{s},\mathbf{d}}^\perp$ *which solves the first equation in system* (6) *and satisfies*

$$\left\| \phi_{\mathbf{z},\mathbf{d}}^\epsilon \right\| \le c|\epsilon|^\eta,$$

*where* $\eta$ *is given in Proposition 3.1.*

Finally, we reduce the problem to a finite dimensional one. We introduce the energy functional $J_\epsilon : \mathfrak{H}_\epsilon \to \mathbb{R}$ defined by

$$J_\epsilon(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega F_\epsilon(u),$$

where

(i) *for problem* $(\mathcal{BN})_\epsilon$

$$F_\epsilon(u) := \frac{1}{p+1} |u|^{p+1} + \frac{1}{2} \epsilon u^2 \quad \text{and } \mathfrak{H}_\epsilon := H_0^1(\Omega)$$

(ii) *for problem* $(\mathcal{AC})_\epsilon$

$$F_\epsilon(u) := \frac{1}{p+1-\epsilon} |u|^{p+1-\epsilon} \quad \text{and } \mathfrak{H}_\epsilon := H_0^1(\Omega) \cap L^{s_\epsilon}(\Omega)$$

(iii) *for problem* $(\mathcal{C})_\epsilon$

$$F_\epsilon(u) := \frac{1}{p+1} |u|^{p+1} \quad \text{and } \mathfrak{H}_\epsilon := H_0^1(\Omega_\epsilon).$$

It is well known that critical points of $J_\epsilon$ are the solutions to problem (4). We introduce the *reduced energy* $\widetilde{J}_\epsilon : \Lambda \to \mathbb{R}$ by

$$\widetilde{J}_\epsilon(\mathbf{z}, \mathbf{d}) := J_\epsilon(V_{\mathbf{z},\mathbf{d}} + \phi_{\mathbf{z},\mathbf{d}}^\epsilon)$$

We prove that critical points of $\widetilde{J}_\epsilon$ generates solutions to the second equation in system (6) and so blowing-up solutions to problem (4).

**Proposition 3.4.** *The function* $V_{\mathbf{z},\mathbf{d}} + \phi_{\mathbf{z},\mathbf{d}}^\epsilon$ *is a critical point of the functional* $J_\epsilon$ *if and only if the point* $(\mathbf{z}, \mathbf{d})$ *is a critical point of the function* $\widetilde{J}_\epsilon$.

3.4. **The reduced problem.** The last step is looking for critical points of the *reduced energy* $\widetilde{J}_\epsilon$.

To to this, we need an accurate asymptotic expansion of the *reduced energy* $\widetilde{J}_\epsilon$,

**Proposition 3.5.** *It holds true that*

$$\widetilde{J}_\epsilon(\mathbf{z}, \mathbf{d}) = a(\epsilon) + b|\epsilon|^\gamma \Phi(\mathbf{z}, \mathbf{d}) + o\left(|\epsilon|^\gamma\right) \tag{7}$$

$C^1$-*uniformly on compact sets of* $\Lambda$. *Here* $a(\epsilon)$ *is constant which depends only on* $n$, $\kappa$ *and* $\epsilon$ *and* $b$ *is a constant. The positive constant* $\gamma$ *and the function* $\Phi$ *are defined as follows.*

*In the case* $\mathfrak{M} - (\mathcal{BN})_\epsilon :$ $\gamma := \frac{n-2}{n-4}$ *and*

$$\Phi(\mathbf{z}, \mathbf{d}) := \begin{cases} \displaystyle\sum_{i=1}^\kappa d_i^{n-2} H(z_i, z_i) + \sum_{i=1}^\kappa \lambda_i \lambda_j \, (d_i d_j)^{\frac{n-2}{2}} \, G(z_i, z_j) - \sum_{i=1}^\kappa d_i^2 & \text{if } \epsilon > 0 \\ \displaystyle\sum_{i=1}^\kappa d_i^{n-2} H(z_i, z_i) + \sum_{i=1}^\kappa \lambda_i \lambda_j \, (d_i d_j)^{\frac{n-2}{2}} \, G(z_i, z_j) + \sum_{i=1}^\kappa d_i^2 & \text{if } \epsilon < 0 \end{cases} \tag{8}$$

*In the case* $\mathfrak{M} - (\mathcal{AC})_\epsilon :$    $\gamma := 1$    *and*

$$\Phi(\mathbf{z}, \mathbf{d}) := \begin{cases} \displaystyle\sum_{i=1}^{\kappa} d_i^{n-2} H(z_i, z_i) + \sum_{i=1}^{\kappa} \lambda_i \lambda_j \left(d_i d_j\right)^{\frac{n-2}{2}} G(z_i, z_j) - \sum_{i=1}^{\kappa} \ln d_i & \text{if } \epsilon > 0 \\ \displaystyle\sum_{i=1}^{\kappa} d_i^{n-2} H(z_i, z_i) + \sum_{i=1}^{\kappa} \lambda_i \lambda_j \left(d_i d_j\right)^{\frac{n-2}{2}} G(z_i, z_j) + \sum_{i=1}^{\kappa} \ln d_i & \text{if } \epsilon < 0 \end{cases} \quad (9)$$

*In the case* $\mathfrak{T} - (\mathcal{AC})_\epsilon :$    $\gamma := 1$    *and*

$$\Psi(\mathbf{z}, \mathbf{d}) := H(z, z) d_1^{n-2} - \sum_{i=1}^{\kappa-1} \frac{1}{(1 + \sigma_i^2)^{\frac{n-2}{2}}} \left(\frac{d_{i+1}}{d_i}\right)^{\frac{n-2}{2}} - \sum_{j=1}^{\kappa} \ln d_i. \quad (10)$$

*In the case* $\mathfrak{T} - (\mathcal{C})_\epsilon :$    $\gamma := \frac{n-2}{2\kappa}$    *and*

$$\Psi(\mathbf{z}, \mathbf{d}) := H(z_0, z_0) d_1^{n-2} + \frac{1}{(1 + \sigma_k^2)^{n-2}} \frac{1}{d_k^{n-2}} + \sum_{j=1}^{\kappa-1} \frac{1}{(1 + \sigma_j^2)^{\frac{n-2}{2}}} \left(\frac{d_{j+1}}{d_j}\right)^{\frac{n-2}{2}}. \quad (11)$$

Finally, we reduce the problem of finding blowing-up solutions to the problem (4) to the problem of finding *good* critical points of the function $\Psi$, which is defined on a finite dimensional space.

**Theorem 3.6.** *Assume* $(\mathbf{z}^*, \mathbf{d}^*) \in \Lambda$ *is a* $C^1-$*stable critical points of the function* $\Psi$. *Then if* $\epsilon$ *is small enough there exists a solution* $u_\epsilon$ *to problem (4) such that*

$$\|u_\epsilon - V_{\mathbf{z}_\epsilon, \mathbf{d}_\epsilon}\| \to 0 \quad as \ \epsilon \to 0$$

*with*

$$(\mathbf{z}_\epsilon, \mathbf{d}_\epsilon) \to (\mathbf{z}^*, \mathbf{d}^*) \quad as \ \epsilon \to 0$$

*Proof.* It follows immediately from the expansion (7) and Proposition 3.4.    □

We remark that $(\mathbf{z}^*, \mathbf{d}^*) \in \Lambda$ is a $C^1-$stable critical points of the function $\Psi$ if

  (i) $(\mathbf{z}^*, \mathbf{d}^*) \in \Lambda$ is an isolated minimum point of $\Psi$,
 (ii) $(\mathbf{z}^*, \mathbf{d}^*) \in \Lambda$ is an isolated maximum point of $\Psi$,
(iii) $(\mathbf{z}^*, \mathbf{d}^*) \in \Lambda$ is a non degenerate critical point of $\Psi$.
 (iv) $(\mathbf{z}^*, \mathbf{d}^*) \in \Lambda$ is a critical point of min-max type of $\Psi$ (according to the Definition given by Del Pino-Felmer-Musso in [21])

## 4. Examples

4.1. **Multi-bubbles for the Brezis-Nirenberg problem** $(\mathcal{BN})_\epsilon$ **and the "slightly sub-critical" problem** $(\mathcal{AC})_\epsilon$ **when** $\epsilon$ **is positive.** According to Theorem 3.6, solutions to these problems are generated by $C^1-$stable critical points of the function $\Psi$ defined in (8) and (9) when $\epsilon > 0$.

  • $\kappa = 1$
    The function $\Psi$ has a minimum point $\Rightarrow$ if $\epsilon \sim 0^+$ problems $(\mathcal{BN})_\epsilon$ and $(\mathcal{AC})_\epsilon$ have a positive solution with one blow-up point.

  • $\kappa = 2$, $\lambda_1 = \lambda_2 = +1$, $\Omega$ *is a dumb-bell with a thin handle*
    The function $\Psi$ has a minimum point $\Rightarrow$ if $\epsilon \sim 0^+$ problems $(\mathcal{BN})_\epsilon$ and $(\mathcal{AC})_\epsilon$ have a positive solution with two different blow-up points.

- $\kappa \geq 2$, $\lambda_1 = \cdots = \lambda_\kappa = +1$, $\Omega$ *is convex*

  The function $\Psi$ does not have any critical points $\Rightarrow$ if $\epsilon \sim 0^+$ problems $(\mathcal{BN})_\epsilon$ and $(\mathcal{AC})_\epsilon$ do not have a positive solution with $\kappa$ different blow-up points.

- $\kappa = 2$, $\lambda_1 = +1$, $\lambda_2 = -1$

  The function $\Psi$ has a minimum point and $(n-1)$ critical points of min-max type $\Rightarrow$ if $\epsilon \sim 0^+$ problems $(\mathcal{BN})_\epsilon$ and $(\mathcal{AC})_\epsilon$ have $n$ sign changing solution with one positive and one negative blow-up points.

4.2. **Multi-bubbles for the Brezis-Nirenberg problem $(\mathcal{BN})_\epsilon$ and the "slightly super-critical" problem $(\mathcal{AC})_\epsilon$ when $\epsilon$ is negative.** According to Theorem 3.6, solutions to these problems are generated by $C^1$−stable critical points of the function $\Psi$ defined in (8) and (9) when $\epsilon < 0$.

- $\kappa = 1$

  The function $\Psi$ does not have any critical points $\Rightarrow$ if $\epsilon \sim 0^-$ problems $(\mathcal{BN})_\epsilon$ and $(\mathcal{AC})_\epsilon$ do not have any positive solution with one blow point.

- $\kappa = 2$, $\lambda_1 = \lambda_2 = +1$ or $\kappa = 3$, $\lambda_1 = \lambda_2 = \lambda_3 = +1$, $\Omega$ *has a hole*

  The function $\Psi$ has a critical point of min-max type $\Rightarrow$ if $\epsilon \sim 0^-$ problems $(\mathcal{BN})_\epsilon$ and $(\mathcal{AC})_\epsilon$ have a positive solution with two different positive blow-up points.

- $\kappa = 2$, $\lambda_1 = +1$, $\lambda_2 = -1$

  The function $\Psi$ does not have any critical points $\Rightarrow$ if $\epsilon \sim 0^-$ problems $(\mathcal{BN})_\epsilon$ and $(\mathcal{AC})_\epsilon$ do not have any sign changing solution with one positive and one negative blow-up points.

4.3. **Tower of bubbles with alternating sign for the "almost critical" problem $(\mathcal{AC})_\epsilon$ when $\epsilon$ is negative.** According to Theorem 3.6, solutions to this problem are generated by $C^1$−stable critical points of the function $\Psi$ defined in (10).

- $\kappa \geq 1$

  The function $\Psi$ has a critical point of min-max type $\Rightarrow$ if $\epsilon \sim 0^+$ problems $(\mathcal{AC})_\epsilon$ has a sign changing solution with $\kappa$ collapsing blow-up points with alternating sign.

4.4. **Tower of bubbles with alternating sign for the Coron's problem $(\mathcal{C})_\epsilon$.** According to Theorem 3.6, solutions to this problem are generated by $C^1$−stable critical points of the function $\Psi$ defined in (11).

- $\kappa \geq 1$

  The function $\Psi$ has a critical point of min-max type $\Rightarrow$ if $\epsilon \sim 0^+$ problems $(\mathcal{C})_\epsilon$ has a sign changing solution with $\kappa$ collapsing blow-up points with alternating sign.

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